Indian Statistical Institute, Bangalore
B. Math. (hons.) Third Year, Second Semester
Differential Equations

Semesteral Examination — Date : 28 April 2023
Maximum marks: 50 — Time: 3 hours

Section I: Answer any four and each question carries 6 marks.

1. Solve $y' + P(x)y = Q(x)y^n$, for $n = 0, 1, 2, \cdots$.
2. Using variation of parameters method find a particular solution of the equation $y'' + y = f(x)$ where f is a continuous function.
3. Let $f(x, y)$ be a continuous function such that $|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$ on a strip defined by $a \leq x \leq b$ and $y \in \mathbb{R}$. If (x_0, y_0) is any point of the strip, then the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has a solution $y = y(x)$ on the interval $[a, b]$.
4. Solve $y' = (1 - x^2)^{\frac{-1}{2}}$ and use it to prove $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\frac{1}{3\times 2^3} + + \frac{1\times 3}{2\times 4}\frac{1}{5\times 2^5} + \cdots$.
5. What is the relation between the function $\sqrt{x}[aJ_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}}) + bJ_{\frac{-1}{3}}(\frac{2}{3}x^{\frac{3}{2}})]$ and the general solution of $y'' + xy = 0$. Justify your answer.
6. Prove that a harmonic function has mean value property.

Section II: Answer any two and each question carries 13 marks.

1. (a) Solve $ydx + (x^2y - x)dy = 0$ using integrating factor (*Marks: 5*).

   (b) Solve $2x^2y'' + x(2x + 1)y' - y = 0$ by Frobenius method.
2. (a) Find all polynomial solutions of $(1 - x^2)y'' - xy' + 4y = 0$.

   (b) Can any polynomial be written as a sum of Legendre polynomials? Justify your answer (*Marks: 7*).
3. (a) State and prove maximum principle for a C^2- function u with $\Delta u \geq 0$.

   (b) Solve $yu_x + xu_y + 2xy = 0$, $u(s, 2s) = 2s^2$ (*Marks: 5*).